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# Topp Leone Exponential – G Family of Distributions: Properties and Application

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**Abstract**

*A new family of distributions called Topp leone Exponential-G family of distributions is developed in this paper. Two sub-models from the family of distributions are also developed; Topp leone Exponential-Exponential (TLEE) and Topp leone Exponential Lomax (TLELx) distributions; their respective density and distribution functions were shown. Some structural properties of this new family of distributions were derived such as moment, moment generating function, probability weighted moment, Renyi entropy and order statistics. The parameters of the model were estimated by using Maximum Likelihood Estimate (MLE) methods. Finally, two real data sets were used to validate the results obtained from MLE. The results showed that TLELx distribution provide a better fit in the data sets than some other existing distributions. Perhaps, this new family of distributions may be used to extend any other existing probability models to modeling positive real data sets.*

## 1. Introduction

In the past decade, many families of univariate distributions have been developed by statisticians through the method of adding parameters to distributions. The likes of Gupta et al. [1]; who pioneered the method of adding parameter to distributions and proposed the exponentiated-G class, also, Marshall and Olkin [2] developed marshall-olkin-G family. In the recent years, Alzaatreh et al. [3] introduced other methods of developing families of distributions. Statisticians have developed new families of univariate distributions through the methods of Alzaatreh et al. [3]. Thus, a new distribution is obtained by extending any of these new families of distributions with some common distributions to get more flexibility in modeling real life data in some areas such as finance, economics, engineering, agriculture, medicine and biological sciences. Although, statisticians aim at developing a new family of univariate distributions in order to generate distributions with left or right-skewed, symmetric or reversed-J shape. To make the kurtosis more flexible when compared to that of any baseline distribution. To construct heavy-tailed distributions for modeling various real data sets. To provide persistently better fits than other generated distributions with the same underlying model.

So many families of univariate distributions have been developed by statisticians with at least one parameter; such as exponentiated generalized-G by Cordeiro et al. [4], new weibull-G by Tahir et al. [5], transmuted exponentiated generalized-G by Yousof et al. [6], kumaraswamy marshall-olkin-G by Alizadeh et al. [7], zografos-balakrishnan odd log-logistic family by Gauss et al. [8], kumaraswamy transmuted-G by Afify et al. [9], burr X-G by Haitham et al. [10], generalized odd generalized exponential family by Alizadeh et al. [11], beta transmuted-H by Afify et al. [12], topp-leone odd log-logistic family by de Brito et al.[13], beta weibull-G by Haitham et al. [14], type I general e xponential class of distributions by Hamedani et al. [15], The transmuted weibull-G family of distributions by Moradet al. [16], The gamma-weibull-G family of distributions by Broderick et al. [17], exponentiated kumaraswamy – G by Ronaldo et al. [18], Burr X Exponential G Family of distributions [19] among others.

In this paper, a new family of univariate distributions with two additional parameters called the Topp leone Exponential-G family (TLE-G) is developed to allow more flexibility in modeling real life data sets, also some of its mathematical properties are studied. The remaining sections of the paper; is organized as follows. The Methodology to the derivation of the new family of distributions is defined in section 2. Properties of TLE-G are discussed in section 3. Parameter estimation is discussed in section 4. Sub models of the TLE-G are discussed in section 5. Applications to two real data sets for the sub-models are shown in Section 6 and Section 7 concludes the paper.

## 2. Methodology

This section presents the method used to derive the cdf and pdf of the new family of distributions called Topp leone Exponential – G family.

### 2.1.Topp Leone – G Family of Distributions

Given the baseline cdf $G(x;\psi)$ and pdf $g(x;\psi)$, with parameter vector $\psi$, Al-Shomrani et al. [20] introduced and developed the Topp Leone – G (TL-G) family of distributions with cdf and pdf given by;

$$F_{TL-G}(x;\sigma,\psi)=\left[G(x;\psi)\right]^{\sigma}\left[2-G(x;\psi)\right]^{\sigma} \tag{1}$$

$$f_{TL-G}(x;\sigma,\psi)=2\sigma g(x;\psi)\left[1-G(x;\psi)\right]\left[G(x;\psi)\right]^{\sigma-1}\left[2-G(x;\psi)\right]^{\sigma-1} \tag{2}$$

respectively for $x>0, \sigma>0$, where $\overline{G}(x;\psi)=1-G(x;\psi)$ and $\psi$ is the vector of parameters for the baseline cdf.

### 2.2. Exponential – G Family of Distributions

Given the baseline cdf $G(x;\beta)$ and pdf $g(x;\beta)$, with parameter vector $\beta$, Faton and Ibrahim [21] suggested the Exponential – G (E-G) family of distributions with cdf and pdf given by,

$$F_{E-G}(x;\lambda,\beta)=1-\exp\left\{-\lambda\left(\frac{G(x,\beta)}{\overline{G}(x,\beta)}\right)\right\} \quad (3)$$

$$f_{E-G}(x;\lambda,\beta)=\frac{\lambda g(x;\beta)}{\left[\overline{G}(x;\beta)\right]^{2}}\exp\left\{-\lambda\left(\frac{G(x,\beta)}{\overline{G}(x,\beta)}\right)\right\} \quad (4)$$

### 2.3.Topp Leone Exponential – G Family of Distributions

The cdf and pdf of the Topp Leone Exponential – G family of distributions (TLE-G) are given by the following lemma:

$$F_{TLE-G}(x;\sigma,\lambda,\beta)=\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\overline{G}(x;\beta)}\right)\right\}\right]^{\sigma} \quad (5)$$

and

$$f_{TLE-G}(x;\beta)=\frac{2\sigma\lambda g(x,\beta)}{\left(\overline{G}(x;\beta)\right)^{2}}\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\overline{G}(x;\beta)}\right)\right\}\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\overline{G}(x;\beta)}\right)\right\}\right]^{\sigma-1} \quad (6)$$

respectively.

**Proof:**

Consider the integration of Equation (2), thus;

$$F_{BXE-G}(t;\sigma,\lambda,\beta)=\int_{0}^{T_{(t)}}2\sigma g(t;\psi)[1-G(t;\psi)]\left[G(t;\psi)\right]^{\sigma-1}\left[2-G(t;\psi)\right]^{\sigma-1}dt \quad (7)$$

where; $T_{(t)}=1-\exp\left\{-\lambda\left(\frac{G(t,\beta)}{\overline{G}(t,\beta)}\right)\right\}$ and

$$dT_{(t)} = \frac{\lambda g(t;\beta)}{\left[\overline{G}(t;\beta)\right]^{2}}\exp\left\{-\lambda\left(\frac{G(t,\beta)}{\overline{G}(t,\beta)}\right)\right\}dt$$
$$= g(t;\psi)\,dt$$

So

$$F_{TLE-G}(t;\sigma,\lambda,\beta)=\int_{0}^{T_{(t)}}2\sigma\left[1-T_{(t)}\right]\left[T_{(t)}\right]^{\sigma-1}\left[2-T_{(t)}\right]^{\sigma-1}dT_{(t)}$$

$$F_{TLE-G}(x;\sigma,\lambda,\beta)=\int_{0}^{T_{(x)}}2\sigma\left[1-T_{(x)}\right]\left[2T_{(x)}-\left(T_{(x)}\right)^{2}\right]^{\sigma-1}dT_{(x)} \quad (8)$$

$$Let\ k = 2T_{(x)} - T_{(x)}^2\ \ then,\ \ dk = 2\left(1 - T_{(x)}\right) dT_{(t)}$$

$$If\ T_{(x)} = 0, then\ k = 0.\ \ Also,\ if\ T_{(x)} = T_{(t)},$$

$$then\ k = 2\left(1 - \exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right) - \left(1 - \exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)^2$$

$$= \int_0^{2\left(1-\exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)-\left(1-\exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)^2} \sigma[k]^{\sigma-1} dk \quad = \left[k^{\sigma}\right]_0^{2\left(1-\exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)-\left(1-\exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)^2}$$

$$= \left[2\left(1-\exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right) - \left(1-\exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)^2 - 0\right]^{\sigma} = \left[\left(2 - 1 - 2\exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\} + 2\exp\left\{-\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\} - \exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)\right]^{\sigma}$$

Therefore,

$$F_{TLE-G}\left(x;\sigma,\lambda,\beta\right) = \left[1 - \exp\left\{-2\lambda\left(\frac{G(x,\beta)}{\bar{G}(x,\beta)}\right)\right\}\right]^{\sigma} \tag{9}$$

Hence the proof of the corresponding cdf of TLE-G family of distributions as defined in Equation (5).

$$When\ x \to 0,\ F_{TLE-G}\left(x;\sigma,\lambda,\beta\right) = \left(1 - \exp\{-2\lambda(0)\}\right)^{\sigma} = 1 - 1 = 0$$

$$when\ x \to \infty, F_{TLE-G}\left(\infty;\sigma,\lambda,\beta\right) = \left[1 - \exp\left\{-2\lambda\left(\infty\right)\right\}\right]^{\sigma} = \left[1 - \exp\left\{-\infty\right\}\right]^{\sigma} \Rightarrow 1$$

And differentiating $F_{TLE-G}\left(\infty;\sigma,\lambda,\beta\right)$ with respect to x will yield Equation (6), which is the probability density function of the TLE-G family of distributions.

Henceforth, a random variable X with density function given in Equation (6) follows $TLE-G\left(x,\psi\right)$, where $\psi = (x;\sigma,\lambda,\beta)$ is a vector of parameters. The survival function S(x, $\psi$ ), the hazard function h(x, $\psi$ ), the reverse hazard function $\Upsilon$ (x, $\psi$ ) and the cumulative hazard function H(x, $\psi$ ) for TLE – G family are given by:

$$S_{TLE-G}\left(x;\psi\right) = 1 - \left[1 - \exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right]^{\sigma} \qquad x\,\varepsilon\,R \tag{10}$$

$$h(x;\psi)=\frac{2\sigma\lambda g(x,\beta)\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right]^{\sigma-1}}{\left(\bar{G}(x;\beta)\right)^{2}\left(1-\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right]^{\sigma}\right)} \tag{11}$$

$$r(x;\psi)=\frac{2\sigma\lambda g(x,\beta)\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right]^{\sigma-1}}{\left(\bar{G}(x;\beta)\right)^{2}\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right]^{\sigma}} \tag{12}$$

$$H(x;\psi)=-\ln\left[1-\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right]^{\sigma}\right] \tag{13}$$

## 3. Properties

This section presents some mathematical properties of the TLE-G family such as incomplete moments, moment generating function, Rényi entropy, order statistics and probability weighted moment.

### 3.1. Linear Representation

In this section, we introduce a useful representation for the TLE-G pdf and cdf. By using generalized binomial and Taylor series expansions on Equation (6). Thus if $|x|\prec 1$ and $k\succ 0$ is a real non integer, the power series holds:

$$(1-x)^{k-1}=\sum_{a=0}^{\infty}\frac{(-1)^{a}\,\Gamma(k)}{a!\,\Gamma(k-a)}x^{a} \tag{14}$$

Applying the idea of Equation (14) on the last term in (6), this becomes;

$$f_{TLE-G}(x;\psi)=\sum_{p=0}^{\infty}(-1)^{p}\frac{\Gamma(\sigma)}{p!\,\Gamma(\sigma-p)}\frac{2\sigma\lambda g(x,\beta)}{\left(\bar{G}(x;\beta)\right)^{2}}\left(\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)^{p+1} \tag{15}$$

Applying the power series to the term $\left(\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)^{p+1}$, Equation (15) becomes:

$$f_{TLE-G}(x;\psi)=\sum_{p,q=0}^{\infty}(-1)^{p+q}\frac{\sigma(2\lambda)^{q+1}\,\Gamma(\sigma)}{p!\,q!\,\Gamma(\sigma-p)}\frac{(p+1)^{q}\,g(x,\beta)\left(G(x;\beta)\right)^{q}}{\left(\bar{G}(x;\beta)\right)^{2+q}} \tag{16}$$

Consider the series expansion in Equation (17), then, apply it to Equation (16) above, this becomes

$$(1-x)^{-k} = \sum_{b=0}^{\infty} \frac{\Gamma(k+b)}{b!\Gamma(k)} x^b, \ |x| \prec 1, b \succ 0 \quad (17)$$

$$f_{TLE-G}(x;\psi) = \sum_{p,q,r=0}^{\infty} (-1)^{p+q} \frac{\sigma(2\lambda)^{q+1}\Gamma(\sigma)(p+1)^q \Gamma\left[(2+q)+r\right]}{p!q!r!\Gamma(\sigma-p)} g(x,\beta)\left(G(x;\beta)\right)^{q+r}$$

$$f_{TLE-G}(x;\psi) = \sum_{p,q,r.=0}^{\infty} \eta_{p,q}\phi_{q+r+1}(x) \quad (18)$$

Where

$$\eta_{p,q} = \frac{(-1)^{p+q}\sigma(2\lambda)^{q+1}\Gamma(\sigma)(p+1)^q\Gamma\left[(2+q)+r\right]}{p!q!r!\Gamma(\sigma-p)(q+r+1)}$$

and $\phi_{q+r+1}(x)=(q+r+1)g(x,\beta)\left(G(x;\beta)\right)^{q+r}$ is the exponentiated-G distribution with power parameter $q+r+1$. By integrating Equation (18) with respect to *x*, we have:

$$F(x;\psi) = \sum_{p,q,r=0}^{\infty} \eta_{p,q}\phi_{q+r+1}(x) \quad (19)$$

Where $\phi_{q+r+1}(x)=G(x)^{q+r+1}$

### 3.2 Raw, Incomplete Moments and Moment Generating Function

Suppose *X* is a random variable with TLE-G distribution, then the raw moment, say $\mu_n^{'}$, is given by

$$\mu_n^{'} = E(x^n) = \int_{-\infty}^{\infty} x^n f_{TLE-G}(x)dx \quad (20)$$

$$= \sum_{p,q,r=0}^{\infty} \eta_{p,q} \int_{-\infty}^{\infty} x^n \phi_{q+r+1}(x)dx$$

$$= \sum_{p,q,r=0}^{\infty} \eta^{*}{}_{p,q}\phi_{n,q+r} \quad (21)$$

Where:

$$\eta^{*}{}_{p,q} = (s+t+1)\eta_{p,q}$$

and

$\phi_{n,q+r} = \int_{-\infty}^{\infty} x^n g(x) G(x)^{q+r}\, dx$ is the probability weighted moment of the baseline distribution. For integer values of *n* and $\mu = \mu_1^{'} = E(x)$, we can obtain the TLE – G central moment of the TLE-G distribution, say $\mu_n$ to be

$$\mu_n = E(X-\mu)^n = \sum_{i=0}^{n} \binom{n}{i} \mu_i^{'} (-\mu)^{n-i} \quad (22)$$

from (22), the measures of skewness and kurtosis of the TLE-G distributions can be obtained as

Skewness (X) $= \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{\left(\mu_2' - \mu_1'^2\right)^{3/2}}$ and Kurtosis (X) $= \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_3' - 3\mu_1'^4}{\mu_2' - \mu_1'^2}$

The rth incomplete moment of X denoted by $m_t(\varpi)$ is

$$m_n = \int_{-\infty}^{\infty} x^n f_{TLE-G}(x)\,dx = \sum_{p,q,r=0}^{\infty} \eta^*{}_{p,q}\phi_{n,q+r}$$

The moment generating function, say $M_x(t)$ of the TLE-G distribution can be obtained as follows:

$$M_x(t) = E\left(e^{tx}\right) = \sum_{p,q,r,v=0}^{\infty} \frac{t^v}{v!}\eta^*{}_{p,q}\phi_{n,q+r} \tag{23}$$

### 3.3.Renyi Entropy

It is used in information theory, queuing theory reliability theory and also applied in statistics. It is useful in indices of diversity and quantifies the uncertainty or randomness of a system. The Renyi Entropy for TLE-G family of distribution is defined by;

$$I_R(v) = (1-v)^{-1}\log\int_{-\infty}^{\infty} f^v(x)\,dx \qquad for\ v \succ 0\ \ and\ \ v \neq 1$$

Consider equation (6) above, thus we have:

$$f^v(x) = \frac{(2\sigma\lambda)^v g(x,\beta)^v}{\left(\bar{G}(x;\beta)\right)^{2v}}\left(\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right)^v\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\bar{G}(x;\beta)}\right)\right\}\right]^{v\sigma-1}$$

After series of expansions by considering Equations (14) and (17); this yields:

$$f^v(x) = \sum_{p,q,r=0}^{\infty} \pi_{p,q}\, g(x)^v\, G(x)^{q+r}$$

Where

$$\pi_{p,q,r} = \frac{\Gamma(v\sigma)(v+p)^q(2\lambda)^q(2\sigma\lambda)^v\,\Gamma\left[(2v+q)+r\right]}{p!q!r!\Gamma(v\sigma-p)\Gamma(2v+q)}$$

Therefore, the Renyi Entropy for the TLE-G family is defined as

$$I_R(v) = (1-v)^{-1}\log\left\{\sum_{p,q,r=0}^{\infty}\pi_{p,q}\int_{-\infty}^{\infty} g(x;\beta)^v\, G(x;\beta)^{q+r}\,dx\right\} \tag{24}$$

### 3.4. Order Statistics

Order statistics play a very significant role in statistics. Let $X_{1;n} \le X_{2;n} \le ... \le X_{n;n}$ be the ordered sample from a continuous population with pdf f(x) and cdf F(x). The pdf of $X_{k;n}$, the $kth$ order statistics is given by;

$$f_{i;n} = \frac{f(x)}{\beta(i, n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-1}{j}F^{j+i-1}(x)$$

From equations (2.31) and (2.32) above, consider their cdf and pdf, thus:

$$f(x)F^{j+i-1}(x)=\frac{2\sigma\lambda g(x,\beta)}{\left(\overline{G}(x;\beta)\right)^2}\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\overline{G}(x;\beta)}\right)\right\}\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\overline{G}(x;\beta)}\right)\right\}\right]^{\sigma-1}$$
$$\times\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\overline{G}(x;\beta)}\right)\right\}\right]^{\sigma(j+i-1)} \quad (25)$$

After series of mathematical expansions by considering Equations (14) and (17) above; Equation (25) becomes:

$$f(x)F^{j+i-1}(x)=\sum_{p,q,r=0}^{\infty}\eta_{p,q}\upsilon_{q+r+1}$$

Consequently, order statistics for the TLE-G family becomes:

$$f_{i;n}=\sum_{j=0}^{n-i}\sum_{p,q,r=0}^{\infty}\eta_{j,p,q}\upsilon_{q+r+1} \quad (26)$$

Where

$$\eta_{j,p,q,r}=\frac{(-1)^{p+q+j}(2\lambda)^{q+1}\sigma\Gamma(n)\Gamma\left[\sigma(j+i)\right]\Gamma\left[(2+q)+r\right](p+1)^q}{j!p!q!r!\Gamma(n-j)\Gamma\left[\sigma(j+i)-p\right]\Gamma\left[(2+q)+r\right](q+r+1)} \text{ and}$$

$$\upsilon_{q+r+1}=(q+r+1)g(x,\beta)\left(G(x;\beta)\right)^{q+r}$$

### 3.5 Probability Weighted Moment

The probability weighted moment (PWM) is a useful approach for estimating the model parameters of that distribution whose inverse form can or cannot be expressed in explicit form.
The $(j+k)^{\text{th}}$ PWM of *X* has the TLE-G distribution, say $M_{j,k}$ is given by;

$$M_{j,k}=E\left[X^jF^k(x)\right] = \int_{-\infty}^{\infty}X^jF(x)^k f(x)dx =$$
$$\int_{-\infty}^{\infty}\left(X^j\frac{2\sigma\lambda g(x,\beta)}{\left(\overline{G}(x;\beta)\right)^2}\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\overline{G}(x;\beta)}\right)\right\}\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\overline{G}(x;\beta)}\right)\right\}\right]^{\sigma-1}\right.$$
$$\left.\times\left[1-\exp\left\{-2\lambda\left(\frac{G(x;\beta)}{\overline{G}(x;\beta)}\right)\right\}\right]^{\sigma k}\right)dx \quad (27)$$

After series of mathematical manipulation, the equation below is obtained

$$M_{j,k}=E\left[X^jF^k(x)\right]=\sum_{p,q,r=0}^{\infty}\pi_{p,q}\zeta_{j,q+r} \quad (28)$$

Where $\pi_{p,q,r}=(-1)^{p+q}\dfrac{(2\lambda)^{q+1}\sigma\Gamma\left[(2+q)+r\right]\Gamma\left[\sigma(k+1)\right](p+1)^q}{p!q!r!\Gamma\left[(2+q)\right]\Gamma\left[\sigma(k+1)-p\right]}$ and

$$\zeta_{j,q+r}(x)=\int_{-\infty}^{\infty}x^jg(x,\beta)\left(G(x;\beta)\right)^{q+r}dx$$

## 4. Parameter Estimation

In the literature, the most useful and commonly used parameter estimation among others; is the maximum likelihood estimate method. Therefore, the maximum likelihood estimators of the unknown parameters of the TLE-G family from complete samples are determined. Let $X_1,...,X_n$ be observed values from TLE – G distributions with vector of parameters $\psi$ . The Log-likelihood function can be expressed as

$$l(\phi) = n\log(2) + n\log(\sigma) + n\log(\lambda) + \sum_{i=1}^{n}\log\left(g\left(x_i;\beta\right)\right) - 2\sum_{i=1}^{n}\log\left(\overline{G}\left(x\right)\right) - 2\lambda\sum_{i=1}^{n}\left(m\left(x;\beta\right)\right)$$
$$+\left(\sigma-1\right)\sum_{i=1}^{n}\log\left[1-\exp\left\{-2\lambda\left(m\left(x;\beta\right)\right)\right\}\right] \tag{29}$$

The component score vector

$$\cup\left(\psi\right) = \frac{\partial\ell}{\partial\psi} = \left(\frac{\partial\ell}{\partial\sigma}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\beta}\right)^T$$

are:

$$\cup_\sigma = \frac{n}{\sigma} + \sum_{i=1}^{n}\log\left[1-\exp\left\{-2\lambda\left(m\left(x;\beta\right)\right)\right\}\right] \tag{30}$$

$$\cup_\lambda = \frac{n}{\lambda} - 2\sum_{i=1}^{n}\left(m\left(x;\beta\right)\right) + \left(\sigma-1\right)\sum_{i=1}^{n}\frac{2\left(m\left(x;\beta\right)\right)\exp\left\{-4\lambda\left(m\left(x;\beta\right)\right)\right\}}{\left[1-\exp\left\{-2\lambda\left(m\left(x;\beta\right)\right)\right\}\right]} \tag{31}$$

and

$$\cup_\beta = \sum_{i=1}^{n}\frac{g'\left(x;\beta\right)}{g\left(x;\beta\right)} - 2\sum_{i=1}^{n}\frac{\overline{G}'\left(x;\beta\right)}{\overline{G}\left(x;\beta\right)} - 2\lambda\sum_{i=1}^{n}m\left(x;\beta\right)m'\left(x;\beta\right) + \left(\alpha-1\right)\sum_{i=1}^{n}\frac{\left[1-\exp\left\{-2\lambda\left(m\left(x;\beta\right)\right)\right\}\right]'}{\left[1-\exp\left\{-2\lambda\left(m\left(x;\beta\right)\right)\right\}\right]} \tag{32}$$

Where:

$$m\left(x;\beta\right) = \frac{G\left(x;\beta\right)}{\overline{G}\left(x;\beta\right)}$$

## 5. The TLE-G Sub Models

Two special sub-models of the TLE-G family, together with the plots of their respective probability density and hazard functions are introduced in this section.

### 5.1 TLE-Exponential (TLEE) Distribution

The cdf and pdf of the Exponential distribution are:

$$G\left(x\right) = 1-\exp\left\{-\gamma x\right\} \tag{33}$$

and

$$g\left(x\right) = \gamma\exp\left\{-\gamma x\right\} \tag{34}$$

$\vartheta, x \succ 0$, respectively. Then the cdf and pdf of TLEE are respectively given by

$$F_{TLE-E}\left(x;\sigma,\lambda,\beta\right) = \left[1-\exp\left\{-2\lambda\left(\exp\left\{\gamma x\right\}-1\right)\right\}\right]^\sigma \tag{35}$$

$$f_{TLE-E}\left(x\right) = \frac{2\sigma\lambda\gamma}{\exp\left\{-\gamma x\right\}}\exp\left\{-2\lambda\left(\exp\left\{\gamma x\right\}-1\right)\right\}\left[1-\exp\left\{-2\lambda\left(\exp\left\{\gamma x\right\}-1\right)\right\}\right]^{\sigma-1} \tag{36}$$

### 5.2. TLE-Lomax (TLELx) Distribution:

The cdf and pdf of the Lomax distribution are

$$G(x)=1-\left[1+\left(\frac{x}{\alpha}\right)\right]^{-\varphi} \tag{37}$$

and

$$g(x)=\left(\frac{\varphi}{\alpha}\right)\left[1+\frac{x}{\alpha}\right]^{-\varphi-1} \tag{38}$$

respectively.

Then, the cdf and pdf of TLE-Lomax are, respectively, given by:

$$F_{TLE-Lx}(x)=\left[1-\exp\left\{-2\lambda\left(\left[1+\left(\frac{x}{\alpha}\right)\right]^{\varphi}-1\right)\right\}\right]^{\sigma} \tag{39}$$

and

$$f_{TLE-Lx}(x)=\frac{2\sigma\lambda\varphi\left[1+\frac{x}{\alpha}\right]^{\varphi-1}}{\alpha}\exp\left\{-2\lambda\left(\left[1+\left(\frac{x}{\alpha}\right)\right]^{\varphi}-1\right)\right\}\left[1-\exp\left\{-2\lambda\left(\left[1+\left(\frac{x}{\alpha}\right)\right]^{\varphi}-1\right)\right\}\right]^{\sigma-1} \tag{40}$$

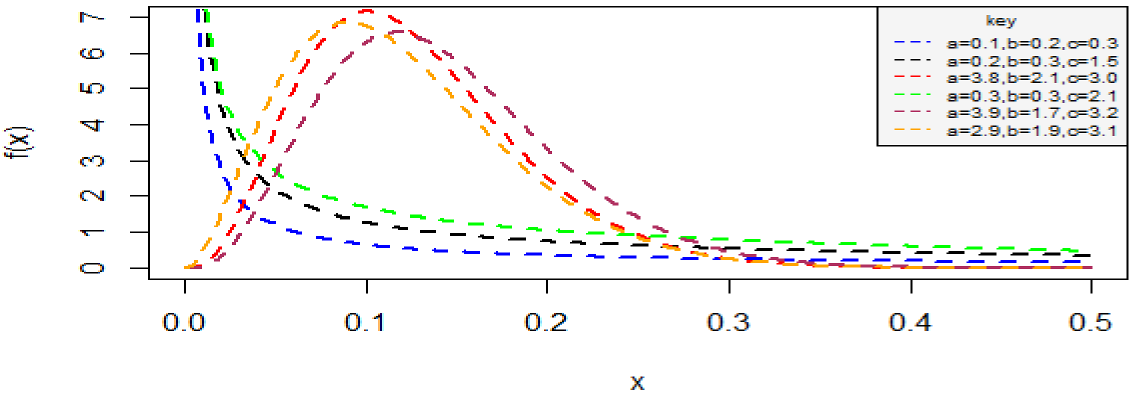


**Figure 1: Plot of the TLE-E density function for some** parameters

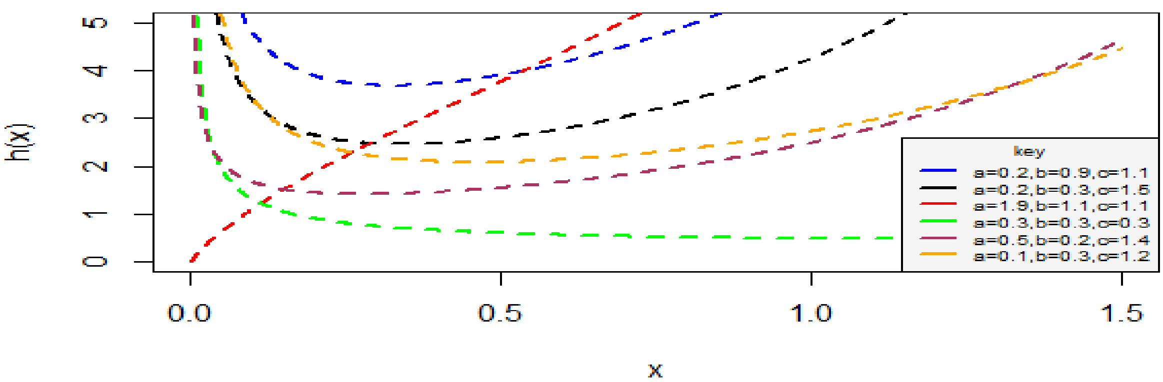


**Figure 2: Plot of the TLE-E hazard function for some parameters**

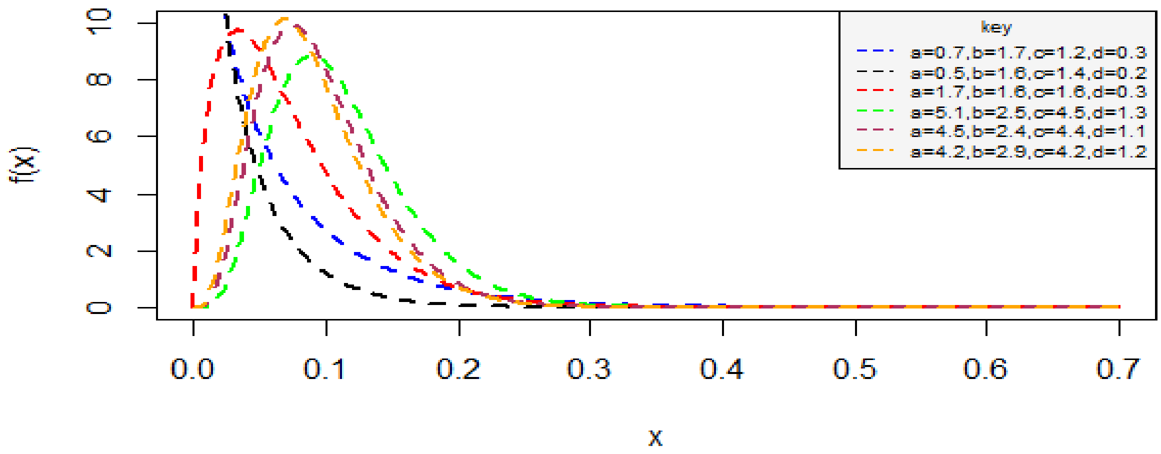


**Figure 3: Plot of the TLE-Lomax density function for some parameters**

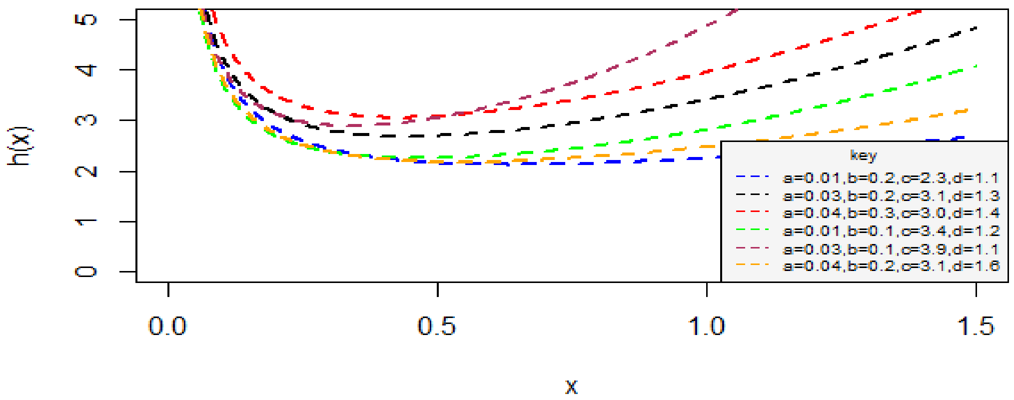


**Figure 4: Plot of the TLE-Lomax hazard function for some parameters**

## 6. Application to Data Sets

In this section, TLELx is considered so as to illustrate its flexibility in application to real life data sets by comparing its performance with other existing distributions. The MLEs and goodness-of-fit statistic for the models' parameters are presented in Tables1 and 2. To compare the fitted models, the paper used some goodness-of-fit measures which include Akaike information criterion (AIC), and consistent Akaike information criterion (CAIC).

We compare the fits of the new TLE-Lx distribution with other competitive models like Top Leone Odd Lindley Lomax (TOLELx), Burr X Lomax (BXLx), Exponential Lomax (ELx) and Lomax (Lx) Distributions. However, their PDFs are available in literature.

**6.1** Application to model the remission times (in months) of bladder cancer patients: This data set represents the remission times (in months) of a random sample of 128 bladder cancer patients previously used by Lee and Wang [22].

**6.2.** Application to model observations of the strengths of 1.5 cm glass fibers: This data has previously been used by Reyad and Othman [23].This was obtained by workers at the UK National Physical laboratory study.Note: These set of data are presented in the Appendix.

The statistical R codes software is used to compute and evaluate MLEs and log – likelihood function. The measure of goodness of fit used are: AIC, CAIC, BIC and HQIC, the lower the values of the criteria the better the fit. The value of TLE-Lx distribution is compared with those of TLOLLx, BXLx, ELx and Lx. Some goodness of fit criteria and MLEs of the models for the first data set and second data set are presented in Tables (1) and (2), respectively.

The values in Tables (1-2) indicate that, the TLE-Lx model has the lowest values for AIC and CAIC among the fitted models (TLELx, TOLLx, BXLx, EXPLx, and Lx) for the first and second real data sets.

The pdfs' and hazard functions' plots are displayed in Figures (1), (2) (3) and (4). It is clear from Figures (1-4) that, the densities of the new distributions (TLEE and TLELx) are unimodal that is, skewed to the right and flat tail with reverse j shape respectively, while the hazard functions' (rate) shapes for the first and second distributions exhibits increasing shape that is upward bathtub shape.

**Table1. MLEs and some Goodness-of-fit criteria for the bladder cancer patients' data**

| Models | Estimate | | | | $-2\hat{\ell}$ | AIC | CAIC |
|---|---|---|---|---|---|---|---|
| TLELx | $\theta$=1.597 | $\lambda$ =11.893 | $\varphi$ =0.150 | $\alpha$ =18.965 | 820.1978 | 828.1978 | 828.5230 |
| TOLLx | $\phi$=1.611 | $\psi$ =5.872 | $\varphi$ =0.263 | $\alpha$ =13.724 | 820.2928 | 828.2929 | 828.6181 |
| BXLx | $\theta$=0.934 | $\varphi$ =0.30 | $\alpha$ =1.020 | | 822.2918 | 828.2918 | 828.4853 |
| EXPLx | $\lambda$ =154.132 | $\varphi$ =0.082 | $\alpha$ =110.130 | | 827.679 | 833.6791 | 833.8726 |
| Lx | $\varphi$ =15.093 | $\alpha$ =131.769 | | | 827.670 | 831.6707 | 831.7667 |

**Table2. MLEs and some Goodness-of-fit criteria for the strengths of 1.5 cm glass fiber's data**

| Models | Estimates | | | | $-2\hat{\ell}$ | AIC | CAIC |
|---|---|---|---|---|---|---|---|
| TLELx | $\theta$=1.46903 | $\lambda$= 0.003 | $\varphi$= 39.320 | $\alpha$ =11.381 | 28.2985 | 36.29858 | 36.9882 |
| BX-Lx | $\theta$=3.044 | $\varphi$=62.765 | $\alpha$ =114.646 | | 35.9748 | 41.9748 | 42.3816 |
| Lx | $\varphi$=120.712 | $\alpha$ =180.980 | | | 178.1588 | 182.1589 | 182.3589 |

## 7. Conclusion

Generating a new family of continuous distributions to extend other well known models; is to improve on the flexibility of the models. A new family of continuous distributions called Topp Leone Exponential–G (TLXE-G) family with additional parameters is developed. More also, new distributions were developed as special cases of the proposed family. Some mathematical properties of this new family of distributions were developed. The flexibility and usefulness of one of the new distributions by studying the remission times (in months) of some random sample of 128 bladder cancer patients and 63 observations of the strengths of 1.5 cm glass fibers obtained by workers at the UK National Physical Laboratory study is demonstrated. It is shown that; the128 bladder cancer patients and the 63 observations of the strengths of 1.5 cm glass fibers sets of data can be modeled using TLELx distribution.

## Conflict of Interests

No competing interest exists.

## Authors' Contributions

"Author A. A Sanusi' designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. 'Author S.I.S Doguwa' restructured the draft and managed the analyses of the study. 'Author I. Audu' reviewed the manuscript and 'Author Y.M Baraya' managed the literature searches. All authors read and approved the final manuscript."

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## Appendix

Data set 1:

0.08, 5.85, 8.26, 11.98, 19.13, 1.76, 10.34, 14.83, 3.88, 5.32, 7.39, 3.25, 4.50,2.09, 3.48, 4.87, 0.81, 2.62,11.64, 17.36, 1.40, 3.02, 4.34, 34.26, 0.90 , 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26,3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 5.41,7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.33, 5.49, 7.66, 11.25, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 17.14, 79.05, 1.35, 2.87, 12.07, 21.73, 2.07, 3.36, 6.93,5.62, 7.87, 3.82, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 14.77, 32.15, 2.64, 5.71, 7.28, 9.74, 14.76, 26.31, 5.32, 7.32, 10.06, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69,4.23, 7.93, 11.79, 18.10, 1.46, 4.40, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 8.65, 12.63, 22.69.

Data set 2:

0.55, 0.74, 0.77, 0.81, 0.84,0.93, 1.04, 1.11,1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50,1.50, 1.51, 1.52, 1.53,1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62,1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68,1.68, 1.69, 1.70, 1.70, 1.73, 1.76,1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.